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RE-ENTRY OF ROTATING MISSILES

by

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AVCO-EVERETT RESEARCH LABORATORY
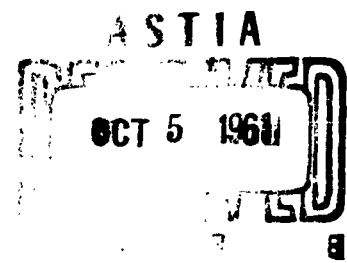a division of
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ABSTRACT

Re-entry of a rotating symmetrical missile, which is assumed to move along a straight path, is examined. The equations of rotational motion are reduced to one second order differential equation for the angle-of-attack, the other angles being then obtainable as quadratures. This form of the equations of motion is suitable for numerical integration, as all of the exact constants of motion have already been integrated out.

Small angle oscillations are considered, and it is shown that previous analyses of the effect of rotation on oscillation are in error, due to an improper procedure for obtaining the small-angle equations. It is pointed out that the precession rate of the oscillational motion is likely to depend as much on non-linearities in the aerodynamic restoring torque as on the rotational velocities.

For a rotating missile with large initial angle-of-attack, the adiabatic invariant is used to calculate the amplitude of oscillation at the lower altitudes.

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I. INTRODUCTION

In this note, we discuss the angular motion of missiles which are rotating as they re-enter the atmosphere, without necessarily making the assumption that the angle-of-attack is small. We do assume that the trajectory of the missile is a straight line, that the missile's external envelope is axially symmetric about one of the principal axes of inertia, and that the center of mass lies on this axis.

II. THE EQUATIONS OF MOTION

We use the coordinate systems shown in Fig. 1. The origin of the coordinates is at the center of gravity of the missile. The missile has moment of inertia A about the X' and Y' axes and moment of inertia C about the Z' axis.

When the angle Θ does not vanish, the aerodynamic forces on the missile lead to a torque whose magnitude $T(\Theta, t)$ depends on Θ (and on the time) and whose direction is parallel to the line ab in Fig. 1, i.e., the line of intersection of the $X'Y'$ plane with the XY plane. Thus, the components of the torque about the Z and Z' axes vanish,

$$T_Z = 0, \tag{1}$$

$$T_{Z'} = 0, \tag{2}$$

and the components about the X' and Y' axes are

$$T_{X'} = -T(\Theta, t) \cos \phi, \tag{3}$$

$$T_{Y'} = T(\Theta, t) \sin \phi. \tag{4}$$

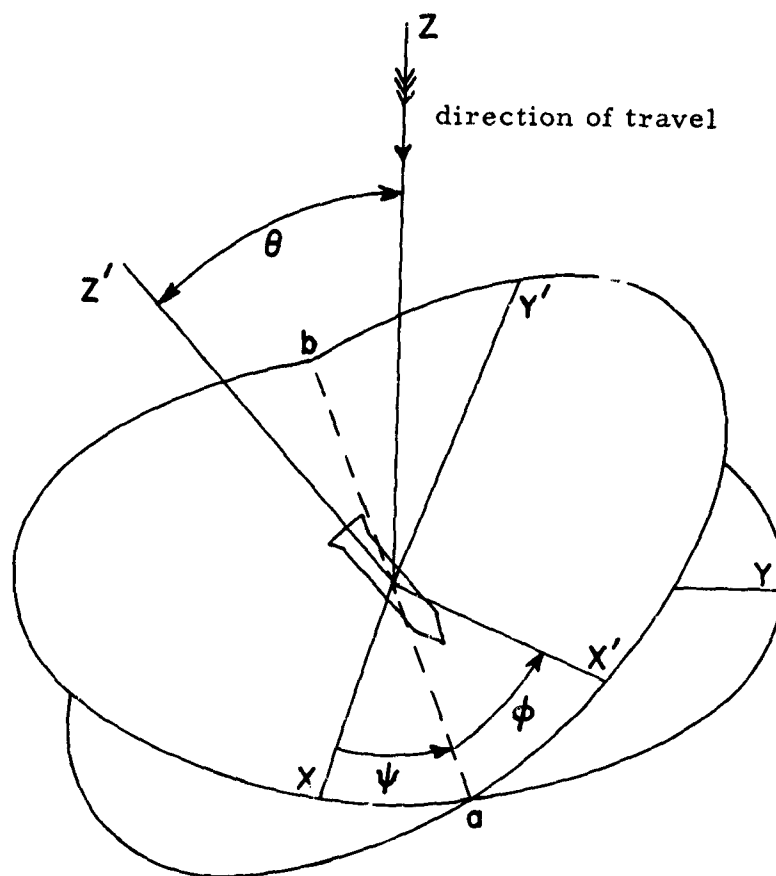The choice of sign is such that positive $T(\Theta)$ implies a torque which tries to reduce Θ.

Fig. 1. Coordinate Systems. The XYZ axes are fixed in space, with the direction of travel of the missile in the minus Z direction. The X'Y'Z' axes are fixed in the missile, and the Z' axis is the symmetry axis. The angles Θ, ψ, and ϕ are the Euler angles.

Let p, q, and r be the angular velocities of the missile about the X', Y' and Z' axes, respectively. Then Euler's Dynamical Equations are:

$$A \frac{dp}{dt} - (A - C) \, qr = - \, T(\Theta, t) \cos \phi \tag{5}$$

$$A \frac{dq}{dt} + (A - C) \, rp = \quad T(\Theta, t) \sin \phi \tag{6}$$

$$C \frac{dr}{dt} \qquad = 0. \tag{7}$$

From Eq. (7), the angular velocity about the symmetry axis is constant,

$$r = \text{constant} \equiv \nu. \tag{8}$$

We shall call ν the spin rate.

Because the torque $T(\Theta, t)$ depends explicitly on the time (on account of the changing air density), the total energy is not a constant of motion. However, multiplying Eq. (5) by p, Eq. (6) by q, and adding, gives

$$\frac{d}{dt} \quad (p^2 + q^2) = - \, \frac{2 \, T(\Theta, t)}{A} \, [\, p \cos \phi - q \sin \phi \,]. \tag{9}$$

At this point it is convenient to introduce the Euler angles instead of p, q, and r. The inter-relations are (dot means time derivative):

$$p = \dot{\psi} \sin \Theta \, \sin \phi + \dot{\Theta} \cos \phi \tag{10}$$

$$q = \dot{\psi} \sin \Theta \, \cos \phi - \dot{\Theta} \sin \phi \tag{11}$$

$$\nu = r = \dot{\phi} + \dot{\psi} \cos \Theta. \tag{12}$$

From Eq. (12) one can obtain the angle ϕ once ψ and Θ are known; but we are not interested in ϕ since it is essentially not observable. However, using Eq.'s (10) and (11) in Eq. (9) gives

$$\frac{d}{dt} \, [\, \sin^2 \Theta \, \dot{\psi}^2 + \dot{\Theta}^2 \,] \quad = - \, \frac{2 \, T(\Theta, t)}{A} \, \dot{\Theta} \tag{13}$$

If $T(\Theta, t)$ did not depend explicity on the time, this equation would immediately yield the total energy constant. The left hand side is the rate of change of the kinetic energy of rotation, less the spin energy, and the right hand side is the rate at which the aerodynamic torque does work (a factor two and the moment of inertia A have been divided out).

Eq. (13) contains two dependent variables, Θ and ψ. The latter can be eliminated by use of another constant of motion, namely, the component of angular momentum about the Z axis. That this component is constant is a result of Eq. (1). From Fig. 1, it is seen that the Z-component L_Z of the angular momentum, expressed in terms of p, q, and r, is

$$L_Z = A \sin \Theta \left[p \sin \phi + q \cos \phi \right] + C r \cos \Theta \qquad (14)$$

Replacing p, q, and r by means of Eq.'s (10) - (12), we find

$$\sin^2 \Theta \, \dot{\psi} + b \nu \cos \Theta = \beta \qquad (15)$$

where we have defined two new constants

$$b \equiv \frac{C}{A} \qquad (16)$$

$$\beta \equiv \frac{L_Z}{A} \qquad (17)$$

Eq. (15) can be used to eliminate $\dot{\psi}$ from Eq. (13), with the result

$$\frac{d}{dt} \dot{\Theta}^2 = - \frac{2T(\Theta, t)}{A} \dot{\Theta} - \frac{d}{dt} \frac{(\beta - b \nu \cos \Theta)^2}{\sin^2 \Theta} \qquad (18)$$

This form of the equation of motion in the Θ variable will be useful for discussing the adiabatic invariant. For the purpose of actually solving the equation it is more convenient to carry out the time differentiations indicated. The equation then takes the form

$$\ddot{\Theta} = - \frac{T(\Theta, t)}{A} - \frac{\partial}{\partial \Theta} \frac{(\beta - b \nu \cos \Theta)^2}{2 \sin^2 \Theta} \qquad (19)$$

This equation has the form of Newton's law for a particle moving in one dimension under a force given by the right hand side. The numerical solution of such equations can be easily accomplished by well known methods.

-4-

Once $\Theta(t)$ has been found from Eq. (19), $\psi(t)$ is found by quadrature from Eq. (15).

There is very little difference between the analysis above and the usual theory of the symmetric top (see, e.g., Osgood, Mechanics, sec. 18). The analysis was presented here to make it clear that the time dependence of the torque does not prevent the reduction of the equations of motion to one second order differential equation in one unknown.

III. SMALL-AMPLITUDE MOTION WITH TIME-INDEPENDENT TORQUE

In this section we shall assume that the torque does not depend explicitly on the time, and that the torque is so large that it dominates the motion to such an extent that Θ moves in a small interval not far from $\Theta = 0$. We approximate the torque by a linear form, letting

$$\frac{T(\Theta)}{A} = \omega_a^2 \, \Theta \qquad (20)$$

The meaning of the constant ω_a can be seen by inserting this form in Eq. (19): ω_a is the frequency of the "pendulum" type of motion that occurs when the angular momentum terms on the right in Eq. (19) are not present. Thus, ω_a may be called the frequency of small oscillations due to aerodynamic forces alone, or simply the aerodynamic frequency. ω_a depends on the air density, the missile velocity, the moment of inertia A, and various aerodynamic coefficients relating to the missile shape. In this section, we regard ω_a as a constant.

We now wish to find the effect of the angular momentum on the frequency. To this end, we expand the second term on the right in Eq. (19) in a power series in Θ, keeping terms up through the first power of Θ. Eq. (19) then becomes

$$\ddot{\Theta} = - \left[\omega_a^2 + \frac{1}{15} (\beta - b\nu)^2 + \frac{1}{4} \beta b\nu \right] \Theta + \frac{(\beta - b\nu)^2}{\Theta^3} \qquad (21)$$

The solutions of this equation are well known. It is, in fact, the equation of motion for the radial coordinate of a two-dimensional harmonic oscillator, with the frequencies equal in the two directions and given by

$$\omega = \sqrt{\omega_a^2 + \frac{1}{15} (\beta - b\nu)^2 + \frac{1}{4} \beta b\nu} \;, \qquad (22)$$

and with angular momentum $(\beta - b\nu)$ about the center. If ξ and η are the (fictitious) Cartesian coordinates of this oscillator, so that

$$\Theta = \sqrt{\xi^2 + \eta^2},$$

the motion in ξ, η makes an ellipse, as indicated in Fig. 2. The frequency ω above is the frequency of traversing the ellipse. The frequency of the Θ motion
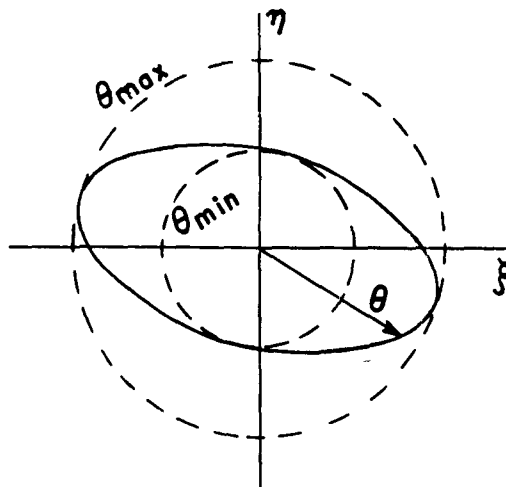

Fig. 2. Fictitious Two-Dimensional Oscillator

is twice ω, but in order to use a definition of frequency which goes over to the commonly accepted definition when there is no angular momentum, we shall call ω the frequency, rather than 2ω.

It must not be concluded that Fig. 2 gives a head-on view of the trajectory of the missile nose. It is the Euler angle ψ, rather than the fictitious angle $\tan^{-1} \eta/\xi$, which determines the angular position of the missile nose when projected into the X, Y plane in Fig. 1. The angle ψ is to be found from Eq. (15),

$$\psi = \int \frac{\beta - b\nu \cos\Theta}{\sin^2 \Theta} \, dt \tag{23}$$

Into the integrand we have to insert Θ as a function of t. From the analogy of Fig. 2, it is easy to see that Θ has the form

-6-

$$\Theta = \sqrt{\lambda + \mu \ \sin 2 \omega t} \tag{24}$$

apart from an arbitrary additive phase in the sine. Here λ and μ are constants. One relation between λ and μ is imposed by the equation of motion, Eq. (21); substituting Eq. (24) therein, one can find that

$$\lambda^2 - \mu^2 = (\beta - b\nu)^2 / \omega^2 \tag{25}$$

This equation is equivalent to the statement that the potential energy, corresponding to the force term in Eq. 21, must have the same value at the two turning points Θ_{min} and Θ_{max} (see Fig. 3). There is no other relation between λ and μ, as the total energy of the Θ motion is arbitrary.

When the expression (24) is substituted for Θ in Eq. (23), it is seen that the integral cannot be performed exactly. However, we may
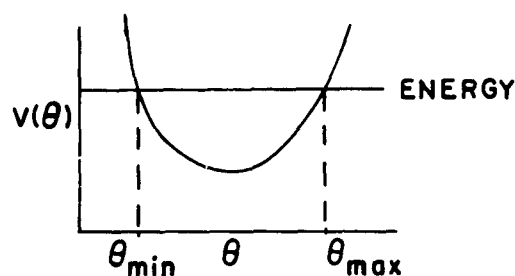

Fig. 3. Energy Diagram
for Θ motion.

expand the integrand in Eq. (23) as a power series in Θ, keeping the same number of terms (three) as we kept in deriving Eq. (21). Eq. (23) then becomes

$$\psi = \int [\frac{\beta - b\nu}{\Theta^2} + \frac{\beta}{3} + \frac{b\nu}{6} + (\frac{\beta}{15} + \frac{7b\nu}{120}) \Theta^2] \ dt \tag{26}$$

The expression (24) is to be inserted for Θ.

Let us compute the increase in ψ during one period of the oscillation, i. e., carry the integration from $t = - \pi/\omega$ to $t = + \pi/\omega$. One then finds (with the help of Eq. 25) that the term $(\beta - b\nu)/\Theta^2$ integrates

to 2π. In fact, this term represents just the angular rate that one would have in the fictitious two-dimensional harmonic oscillator. Thus, for this term alone, Fig. 2 would give the correct head-on view. However, the other terms in the integrand of Eq. (26) lead to an <u>average</u> <u>rate</u> of advance of the angle of aphelion of

$$\frac{\Delta \psi}{\Delta t}\bigg)_{ap} = \frac{\beta}{3} + \frac{b\nu}{6} + \left(\frac{\beta}{15} + \frac{7\,b\,\nu}{120}\right)\bar{\Theta}^2 \tag{27}$$

where $\bar{\Theta}^2 = \lambda$ is the time average of Θ^2.

Because of the expansions used above, all the results of this section are valid only when

$$\bar{\Theta}^2 \ll 1 \tag{28}$$

By calculating the angle Θ at which the force term in Eq. (21), i.e., the right hand side, vanishes, one sees that the small angle requirement demands generally that

$$\omega_a^2 \gg \beta^2$$
$$\tag{29}$$
$$\omega_a^2 \gg b^2 \nu^2$$

Thus, generally the results of this section are <u>valid</u> <u>only</u> <u>when</u> the aerodynamic frequency is much greater than the <u>angular</u> <u>velocities</u> of rotation.

The only exception to this rule is the case when β is very nearly equal to $b\nu$, in which case we do not need large ω_a in order to have small $\bar{\Theta}^2$. A spinning object with little wobble (i.e., with most of its angular momentum about the symmetry axis), and with a small angle Θ between the symmetry axis and the Z axis, has β very nearly equal to $b\nu$. Thus, this is the case of a gyroscope (i.e., no torque) nearly aligned and only slightly wobbling. For this case, Eq. (22) gives the frequency $\omega = b\nu/2$ and Eq. (27) gives the precession rate $= b\nu/2$. Both of these results are correct for the slightly wobbling gyroscope.

The corrections to the frequency due to rotation, and the precession of the aphelion, differ from expressions given for these effects in the literature.[1,2] The reason for this is that the small angle approximation, in the references cited, was not made to sufficient accuracy to get the frequency correction and the precession rate correctly. In these references, the approximation made was to replace cos Θ by unity and sin Θ by Θ. However, in arriving at the expansion in Eq. (21), it was

-8-

necessary to take the first three terms in the expansion of the sines and cosines (in the term arising from rotation) in order to get the first order correction to the frequency.

Let us estimate the error induced by neglecting higher powers of Θ on the right hand side of Eq. (21). The rotational term in Eq. (19) (i. e., the second term on the right) will contribute a term in Θ^3. The coefficient of this term will be a linear combination of β^2, $b^2 \nu^2$, and $\beta b \nu$. The torque function $T(\Theta)$ is likely to be non-linear. If we assume that $T(\Theta)$ is expandable as a power series and is an odd function of Θ, the next term in $T(\Theta)$ would also be of order Θ^3. Because the torque function is likely to be quite non-linear, the coefficient of the Θ^3 term is likely to be as big as the coefficient of the linear torque term, namely ω_a^2. Thus, we should consider the effect of adding a term

$$K \Theta^3$$

to the right hand side of Eq. (21), where

$$K = K_{rot} = O(\beta^2, \ b^2 \nu^2, \ \beta b \nu)$$

for the rotational corrections, and

$$K = K_T = O(\omega_a^2)$$

for the non-linear torque corrections. The latter correction is apt to be much the larger, since

$$\Theta^4 \approx O \left(\frac{(\beta - b\nu)^2}{\omega_a^2} \right),$$

according to Eq. (21), and we require Θ to be small. Now, adding a term $K \Theta^3$ to the right hand side of Eq. (21) will perturb its solution by $O(K\Theta^3/\omega_a^2)$; that is, if $\Theta_0(t)$ and $\Theta_1(t)$ are the unperturbed and perturbed solutions, respectively,

$$\Theta_1(t) = \Theta_0(t) \left[1 + O \left(\frac{K \Theta_0^2}{\omega_a^2} \right) \right]$$

Thus, the fractional correction to $\Theta_0(t)$ is of order Θ_0^6 if $K = K_{rot}$, but of order Θ_0^2 if $K = K_T$. It can then be seen from Eq. (26) that the non-linear torque correction gives a correction to the rate of advance of aphelion which is of the same order as the rate previously calculated, Eq. (27). Thus, a non-linear torque can completely modify the precession of the aphelion angle.

-9-

IV. SMALL-AMPLITUDE MOTION FOR EXPONENTIALLY VARYING ω_a

We again assume that the motion is confined to small angles, but let the aerodynamic frequency vary with time according to the approximation

$$\omega_a^2 = \omega_0^2 \ e^{\alpha t} \tag{30}$$

We shall call the constant α the "density rate". It is related to the missile velocity, the re-entry angle and the atmospheric scale height. A typical value for α is about $1/4$ sec^{-1}.

Equation (21) covers this case, since ω_a^2 was not differentiated in deriving it. Let us re-write Eq. (21) in the form

$$\ddot{\Theta} = - [\omega_0^2 \ e^{\alpha t} + \omega_1^2] \ \Theta + \frac{L^2}{\Theta^3} \tag{31}$$

where we have defined

$$\omega_1^2 \equiv 1/15 \ (\beta - b\nu)^2 + 1/4 \ \beta b\nu \tag{32}$$

and

$$L = \beta - b\nu \tag{33}$$

It can be shown that ω_1^2 is positive.

Again, Eq. (31) is the equation of motion for the radial coordinate of a fictitious harmonic oscillator in two dimensions, but with the spring constant varying with time. If ξ and η are Cartesian coordinates for this fictitious oscillator, they satisfy equations of motion of the form

$$\ddot{\xi} = - [\omega_0^2 \ e^{\alpha t} + \omega_1^2] \ \xi \tag{34}$$

The substitution

$$y = e^{\frac{\alpha t}{2}} \tag{35}$$

converts Eq. (34) to

$$\frac{1}{y} \frac{d}{dy} \ y \frac{d}{dy} \ \xi + [\ 4 \ \frac{\omega_0^2}{\alpha^2} + 4 \ \frac{\omega_1^2}{\alpha^2} \ \frac{1}{y^2}] \ \xi = 0 \tag{36}$$

-10-

This equation has the solutions

$$\xi = J_{\pm ik} \left(\frac{2 \omega_0}{a} \, y \right) \tag{37}$$

where

$$k = \frac{2\omega_1}{a} \tag{38}$$

and J_{ik} is the Bessel function of imaginary order

$$J_{ik} \left(\frac{2 \omega_0}{a} \, y \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \, (n+ik)!} \left(\frac{\omega_0 \, y}{a} \right)^{2n+ik} \tag{39}$$

The fictitious coordinates ξ and η are each a linear combination of J_{ik} and J_{-ik}, and Θ is then given by

$$\Theta = \sqrt{\xi^2 + \eta^2} \tag{40}$$

That the Bessel functions enter the solution of the small angle problem is well known. In fact, the principle difference between these results and those of reference 1 is that the frequency ω_1 is different.

For early enough times, the variable y is small, and only the first term in the series (39) need be considered. Then, apart from constant factors, the solutions (37) are

$$\xi = y^{\pm ik} = \left(e^{\frac{at}{2}} \right)^{\pm i \frac{2 \omega_1}{a}} = e^{\pm i \omega_1 t} \tag{41}$$

Of course, the early time solution is only valid provided the initial "gyroscopic" motion is confined to small angles, as was discussed in the previous section. For this case, $\omega_1 = b\nu/2$, and Eq. (41) is correct.

It is also worth noting at what time the aerodynamic forces begin seriously to change the motion from its initial "gyroscopic" form. Taking the first two terms of the series (39) leads to, apart from a constant factor

-11-

$$\xi = e^{i\omega_1 t} \left[1 - \frac{1}{1 + ik} \left(\frac{\omega_a}{a} \right)^2 \right] \tag{42}$$

where we have used the fact that

$$\omega_0 \, y = \omega_a \tag{43}$$

Of course, ω_a depends on the time. We now examine two cases.

Case 1. $\omega_1 \ll a$, or rotation rate small compared to density rate. In this case k is small compared to unity, and the amplitude of the motion is beginning to be seriously changed when

$$\omega_a \approx a \qquad (\omega_1 \ll a) \tag{44}$$

Case 2. $\omega_1 \gg a$, or rotation rate large compared to density rate. In this case, k is large, and the _phase_ of the motion begins to be seriously changed when

$$\omega_a^2 \approx a^2 k \approx 2\omega_1 a \quad (\text{phase}, \ \omega_1 \gg a) \tag{45}$$

However, the _amplitude_ of the motion is not yet seriously changed at this point. This results from the fact that, for large k, the terms in the series (39) up to $n \approx k$ approximate the series for the exponential

$$e^{i\omega_1 t} \qquad \exp \left(\frac{i\omega_a^2}{a^2 k} \right)$$

If the n is dropped in the factors $(n + ik)$, one gets exactly this result, which has no change in amplitude. The correction is of order

$$\frac{1}{k^2} \left(\frac{\omega_a}{a} \right)^2 \ ,$$

and the amplitude of the motion is seriously changed when

$$\omega_a \approx \alpha k \approx 2\omega_1 \qquad (\text{amplitude, } \omega_1 \gg \alpha) \qquad (46)$$

We may combine the results (44) and (46) into the statement that the amplitude is seriously changed when the air density is high enough to make

$$\omega_a^2 = \alpha^2 + 4 \omega_1^2 \qquad (\text{amplitude}) \qquad (47)$$

Finally, let us examine the motion at late times, when ω_a is large compared to ω_1 and α. The asymptotic form of the Bessel function leads to

$$J_{ik} \rightarrow \sqrt{\frac{\alpha}{\pi \omega_a}} \left[1 - \frac{\omega_1^2 + \frac{1}{16}\alpha^2}{4\omega_a^2} \right] \cos\left\{ \frac{2\omega_a}{\alpha} - \frac{\omega_1^2 + \frac{1}{16}\alpha^2}{\omega_a \alpha} \right.$$

$$\left. - \frac{i\pi\omega_1}{\alpha} - \frac{\pi}{4} \right\} \qquad (48)$$

Remember that ω_a increases exponentially with time,

$$\frac{d\omega_a}{dt} = \frac{\alpha}{2} \omega_a \qquad (49)$$

The "instantaneous frequency" of the motion is the rate at which the argument of the cosine in Eq. (48) increases, or

$$\text{"instant. } \omega\text{"} = \omega_a + \frac{1}{2} \frac{\omega_1^2 + \frac{1}{16}\alpha^2}{\omega_a} \qquad (50)$$

Except for the term in α^2, this frequency is the same as that given by Eq. (22), provided the radical there is expanded for large ω_a. (Remember the definition (32)). We may say that the α^2 term is an effect of the changing "spring constant".

We also see from Eq. (48) that the amplitude of the oscillations falls off as

$$\text{amplitude} \sim \frac{1}{\sqrt{\omega_a}} \sim \frac{1}{(\rho_{air})^{\frac{1}{4}}} \tag{51}$$

In fact, by comparing the initial and asymptotic forms of the Bessel function, one can find the ratio of the amplitude to the initial amplitude

$$\frac{\text{amplitude}}{\text{initial amplitude}} \approx \sqrt{\frac{a}{\pi \omega_a}} \ \cosh\left(\frac{\pi \omega_1}{a}\right) \bigg/ \ |\,(ik)!\,|$$

$$\approx \sqrt{\frac{2\omega_1}{\omega_a}} \cosh\left(\frac{\pi\omega_1}{a}\right) \bigg/ \left[\sinh\left(\frac{2\pi\omega_1}{a}\right)\right]^{1/2} \tag{52}$$

(See Jahnke and Emde, formula f, p. 11, for the factorial).

Again, it should be borne in mind that this ratio is accurate only if the initial gyroscopic motion is confined to small angles.

V. DISCUSSION OF THE LARGE ANGLE CASE BY USE OF THE ADIA-BATIC INVARIANT.

The results of Sections III and IV depend on the assumption that the angle Θ is small. This angle does become small eventually, deep in the atmosphere, but it can be large initially. In this section we shall use the adiabatic invariant to discuss the large angle case. This treatment will depend on the assumption that the initial rotational rate is large compared to the density rate a.

We return to the exact equations, Eq. (18) or (19). Initially, outside the atmosphere, the torque is negligible. In this case, Eq. (18) can be integrated once, giving

$$\dot{\Theta}^2 + \frac{(\beta - b\nu\cos\Theta)^2}{\sin^2\Theta} = \text{constant} \equiv h \tag{53}$$

-14-

The constant h is the initial value of $p^2 + q_1^2$ (see Eq. (9)), the square of the angular velocity about the two axes in the plane perpendicular to the symmetry axis. $\sqrt{h}$ could be called the "tumble" angular velocity, while ν is the spin angular velocity. h can have any non-negative value, independent of ν.

Eq. (53) can be simplified by the substitution

$$\cos \Theta = u, \tag{54}$$

which leads to

$$\dot{u}^2 = h(1-u^2) + (\beta - b\nu u)^2 \tag{55}$$

This is the energy equation for a particle moving in a quadratic potential, and therefore has harmonic solutions. The frequency Ω_0 of the oscillations in u, and Θ, is given by the coefficient of the quadratic term in u, and is

$$\Omega_0 = b^2 \nu^2 + h \tag{56}$$
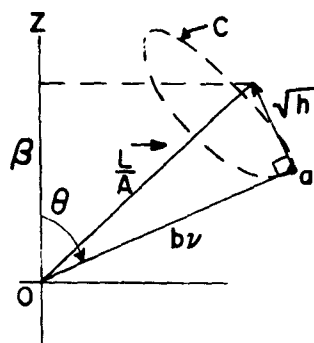
Fig. 3. Relation of Various Parameters.

Fig. 3 is a vector diagram of the angular momenta, divided by the moment of inertia A, for the present case of negligible torque. The total angular momentum $\vec{L}$ is constant in magnitude and direction. The constant β is the projection of $\vec{L}/A$ onto the Z-axis. $b\nu$ is the angular momentum, divided by A, about the symmetry axis, and $\sqrt{h}$ is the component of angular momentum, divided by A, perpendicular to the symmetry axis. The apex a of the right triangle formed by $\vec{L}/A$, $b\nu$, and $\sqrt{h}$ revolves around the circle C with angular velocity Ω_0.

Now let the missile begin to enter the atmosphere. The torque term in Eq. (19) builds up exponentially like e^{at} . But if

$$\Omega_o \gg a, \tag{57}$$

the change in the force function in Eq. (19) per cycle $2\pi/\Omega_o$ will be small, and the action I,

$$I \equiv \int_{\text{cycle}} \dot{\Theta} \, d\Theta = \int_{\text{cycle}} \dot{\Theta}^2 \, dt, \tag{58}$$

will be approximately constant. We shall evaluate this integral shortly.

First, however, note that at some altitude, which we shall call the critical altitude, the torque term in Eq. (19) will become of about equal importance to the rotational term. At the critical altitude, the aerodynamic frequency ω_a is about equal to $\Omega_o/2$, and the motion has just become seriously changed by the aerodynamic forces. Above the critical altitude, the frequency of the Θ variations is close to Ω_o, while below the critical altitude the frequency increases, following and approaching $2\omega_a$. Thus, if the adiabatic condition is satisfied initially, Eq. (57), it will be satisfied at all times, since the density rate a is (approximately) constant. The action will therefore be approximately constant at all times, both above and below the critical altitude. Using this fact, we can obtain a relation between the initial rotation rates, above the critical altitude, and the amplitude of the small oscillations at altitudes somewhat lower than the critical altitude. To do this we have to evaluate the action integral for the initial gyroscopic motion and for the asymptotic small amplitude motion.

From Eq. (53), the initial action is

$$I_{in} = \int_{\text{cycle}} \dot{\Theta} \, d\Theta = 2 \int_{\Theta_1}^{\Theta_2} d\Theta \sqrt{h - \frac{(\beta - b\nu \cos \Theta)^2}{\sin^2 \Theta}} \tag{59}$$

where Θ_1 and Θ_2 are the two zeroes of the radical. The integral can be simplified by changing the variable to $u = \cos \Theta$ and can be evaluated with the help of formula (380. 311) of Dwight, p. 71. The result is

$$I_{in} = 2\pi \left[\Omega_o - \mathcal{L}(|\beta|, |b\nu|) \right] \tag{60}$$

where $\mathcal{L}(|\beta|, |b\nu|)$ means the larger of $|\beta|$ and $|b\nu|$. (This choice corresponds to whether the tip of the missile, in its gyroscopic motion, does or does not encircle the Z-axis.)

At altitudes somewhat lower than the critical altitude, the aerodynamic forces dominate the motion, forcing it to small angles. The form of motion is then given approximately by the results of Sec. III, in particular by Eq.'s (24) and (25). Thus the final action is approximately

$$I_{fin} = \int_{cycle} \dot{\Theta}^2 \, dt = 2 \int_{t=-\pi/4\omega}^{+\pi/4\omega} \frac{4\mu^2 \omega^2 \cos^2(2\omega t)}{\lambda + \mu \sin(2\omega t)} \, dt \tag{61}$$

Again, the substitution $u = \sin(2\omega t)$ and the same formulae in Dwight, enable one to evaluate this integral, with the result

$$I_{fin} = \pi \omega \left[\lambda - \sqrt{\lambda^2 - \mu^2} \right]$$

Using Eq. (25) and the fact that $\overline{\Theta^2} = \lambda$, we rewrite this equation as

$$I_{fin} = \pi \omega \overline{\Theta^2} - \pi |\beta - b\nu| \tag{62}$$

Equating I_{in} and I_{fin}, we obtain an equation expressing the mean-square angle of attack (at altitudes somewhat less than the critical altitude) in terms of the initial rotation rates

$$\overline{\Theta^2} = \frac{1}{\omega} \left\{ |\beta - b\nu| + 2\Omega_o - 2\mathcal{L}(|\beta|, |b\nu|) \right\} \tag{63}$$

For ω here, one may use Eq. (22), or simply replace ω by ω_a, since the corrections are small at later times. Eq. (63) shows that the mean-

-17-

square angle is larger for larger initial rotation rates.

We may ask whether Eq. (63) agrees with the result of Sec. IV in the case where the initial gyroscopic motion is confined to small angles. In this case, β, $b\nu$, and Ω_o are all approximately equal, differing only by about $\Theta_o^2 b\nu/2$, where Θ_o is representative of the initial small angles. Equation (63) then becomes, within a factor of two or so

$$\overline{\theta^2} = \Theta_o^2 \frac{b\nu}{\omega_a} \tag{64}$$

This is to be compared with Eq. (52), evaluated for the case where ω_1/a is large, and $\omega_1 = b\nu/2$ (see Eq. (32)) since $\beta \approx b\nu$. This evaluation leads to the same result (64).

VI.   THE CASE OF LARGE INITIAL ANGLES WITH SLOW ROTATION

The only case not covered in previous sections is that where the initial rotation rates are small compared to the density rate a but initial angles are large. We shall now give an approximate discussion of this case.

If the rotation rates are small compared to a, they may be ignored entirely, except insofar as they determine the angle of attack the missile will have when it arrives at the altitude where the aerodynamic forces first become appreciable. Thus we may consider Eq. (19) without the rotational term. Let us consider the torque function $T(\Theta)$. It will have a shape something like that indicated in Fig. 4. Up to an angle $\Theta \approx \pi/2$, a linear fit gives a fairly decent approximation. Therefore, if the initial angle is not greater than about $\pi/2$, we may use the "small-angle" result (52) evaluated for the case where ω_1/a is small, namely

$$\frac{\text{amplitude}}{\text{initial angle}} \approx \sqrt{\frac{a}{\pi \omega_a}} \tag{65}$$

If the initial angle is larger than about $\pi/2$, the amplitude (at the lower altitudes) will be larger than this formula implies, because the first swing round of the missile will take longer than it would if the torque continued to follow the linear fit. In fact, if the initial angle is almost π, it will take a very long time for the first swing round. However, such
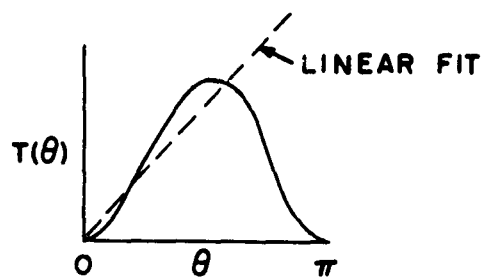
-18-

cases are fairly improbable.

Fig. 4

To get further information, one can resort to numerical solution of Eq. (19).

ACKNOWLEDGMENTS

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-20-

Avco-Everett Research Laboratory, Everett, Massachusetts
RE-ENTRY OF ROTATING MISSILES, by Conrad L. Longmire.
December, 1960. 20 p. incl. illus. (Avco-Everett Research Note 213;
AFBMD TR-61-10)
(Contract AF 04(647)-278 and
Purchase Order Number D-603009;
Prime Contract DA-30-069-ORD-1955)   Unclassified report

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Division, Air Research and Development Command and Bell Telephone
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Re-entry of a rotating symmetrical missile, which is assumed to move
along a straight path, is examined. The equations of rotational motion
are reduced to one second order differential equation for the angle-of-
attack, the other angles being then obtainable as quadratures. This form
of the equations of motion is suitable for numerical integration, as all of
the exact constants of motion have already been integrated out. Small

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     Prime Contract DA-30-069-ORD-1955.

angle oscillations are considered, and it is shown that previous analyses
of the effect of rotation on oscillation are in error, due to an   oper
procedure for obtaining the small-angle equations. It is pointed out that
the precession rate of the oscillational motion is likely to depend as much
on non-linearities in the aerodynamic restoring torque as on the rotational
velocities. For a rotating missile with large initial angle-of-attack, the
adiabatic invariant is used to calculate the amplitude of oscillation at the
lower altitudes.